A New Characterization of the Clifford Torus

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Abstract

In this paper we introduce the notion of contact angle for an immersed surface in three dimensional sphere. We deduce formulas for the Laplacian and for the Gaussian curvature, and we classify minimal surfaces in S^3 with constant contact angle. Also, we give an example of a minimal surface in S^3 with non constant contact angle.

Keywords: contact angle,contact distribution, Clifford torus, minimal surfaces.

2000 Math Subject Classification: 53C42 - 53D10 - 53D35.

1 Introduction

The notion of Kähler angle was introduced by Chern and Wolfson in [4] and [12]; it is a fundamental invariant for minimal surfaces in complex manifolds. Using the technique

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of moving frames, Wolfson obtained equations for the Laplacian and Gaussian curvature for an immersed minimal surface in $\mathbb{CP}^n$. Later, Kenmotsu in [8], Ohnita in [10] and Ogata in [11] classified minimal surfaces with constant Gaussian curvature and constant Kähler angle.

A few years ago, Li in [14] gave a counterexample to the conjecture of Bolton, Jensen and Rigoli (see [3]), according to which a minimal immersion (non-holomorphic, non anti-holomorphic, non totally real) of a two-sphere in $\mathbb{CP}^n$ with constant Kähler angle would have constant Gaussian curvature.

In [9] we introduced the notion of contact angle, that can be considered as a new geometric invariant useful to investigate the geometry of immersed surfaces in contact riemannian manifolds. Geometrically, the contact angle (β) is the complementary angle between the contact distribution and the tangent space of the surface. Also in [9], we deduced formulas for the Gaussian curvature and the Laplacian of an immersed minimal surface in S^5, and we found a two parameters family of minimal torus in S^5 with constant contact angle.

In this paper, we will construct two characterizations of the Clifford torus in S^3 using the contact angle. We obtain the following formula for the Gaussian curvature of an immersed minimal surface in S^3:

$$K \;=\; 1 - |\nabla\beta + e_1|^2$$

Also, we obtain the following equation for the Laplacian:

$$\Delta(\beta) \;=\; -\tan(\beta)|\nabla\beta + 2e_1|^2$$

where e_1 is the characteristic field defined in section 2 and introduced by Bennequin in [1].

Using the equations of Gauss and Codazzi, we have proved the following two theorems:

Theorem 1. *The Clifford torus is the only minimal surface in S^3 with constant contact angle.*

Theorem 2. *The Clifford Torus is the only minimal surface in S^3 with contact angle $0 \leq \beta < \frac{\pi}{2}$ (or $-\frac{\pi}{2} < \beta \leq 0$)*

At the last section, we give two examples of minimal surfaces in S^3. At the first one, we determine that the contact angle (β) of the Clifford torus is ($\beta = 0$) and the second one we determine that the contact angle of the totally geodesic sphere is ($\beta = arccos(x_2)$), and therefore, non constant (see section 6).

2 Contact Angle for Immersed Surfaces in S^3

Consider in $\mathbb{C}^2$ the following objects:

- the Hermitian product: $(z,w) = z^1\bar{w}^1 + z^2\bar{w}^2$;
- the inner product: $\langle z,w\rangle = Re(z,w)$;
- the unit sphere: $S^3 = \{z \in \mathbb{C}^2 | (z,z) = 1\}$;
- the *Reeb* vector field in S^3, given by: $\xi(z) = iz$;
- the contact distribution in S^3, which is orthogonal to ξ:

$$\Delta_z = \{v \in T_z S^3 | \langle \xi, v\rangle = 0\}.$$

We observe that Δ is invariant by the complex structure of $\mathbb{C}^2$.

Let now S be an immersed orientable surface in S^3.

Definition 1. *The* contact angle β *is the complementary angle between the contact distribution* Δ *and the tangent space* TS *of the surface.*

Let (e_1, e_2) be a local frame of TS, where $e_1 \in TS \cap \Delta$. Then $\cos\beta = \langle \xi, e_2\rangle$. In S^3 consider the frame $f_1 = z^\perp$, $f_2 = iz^\perp$ and $f_3 = iz$.
The covariant derivative is given by:

$$\begin{array}{rcl} Df_1 & = & w_1^2\, f_2 + w^2\, f_3 \\ Df_2 & = & w_2^1\, f_1 - w^1\, f_3 \\ Df_3 & = & -w^2\, f_1 + w^1\, f_2 \end{array} \tag{1}$$

where (w^1, w^2, w^3) is dual frame of (f_1, f_2, f_3).
Consider e_1 unitary vector field in $TS \cap \Delta$, where Δ is the contact distribution.
Then we have:

$$\begin{array}{rcl} e_1 & = & f_1 \\ e_2 & = & \sin(\beta)\, f_2 + \cos(\beta)\, f_3 \\ e_3 & = & -\cos(\beta)\, f_2 + \sin(\beta)\, f_3 \end{array} \tag{2}$$

where β is the angle between f_3 and e_2, (e_1, e_2) are tangents to S and e_3 is normal to S

3 Equations for the Curvature and Laplacian

In this section, we deduce the equations for the Gaussian curvature and for the Laplacian of a minimal surface in S^3 in terms of the contact angle.
Consider $(\theta^1, \theta^2, \theta^3)$ dual frame of (e_1, e_2, e_3)

$$\begin{array}{rcl} \theta^1 & = & w_1 \\ \theta^2 & = & \sin(\beta)\, w_2 + \cos(\beta)\, w_3 \\ \theta^3 & = & -\cos(\beta)\, w_2 + \sin(\beta)\, w_3 \end{array} \tag{3}$$

At the surface S, we have $\theta^3 = 0$, then we obtain the equation:

$$sin(\beta)\, w^3 \;\; = \;\; cos(\beta) w^2 \tag{4}$$

we have also

$$\begin{aligned} w^2 &= \sin\beta\theta^2 \\ w^3 &= \cos\beta\theta^2 \end{aligned}$$

It follows from (3) that:

$$\begin{aligned} d\theta^1 &+ \sin(\beta)(w_2^1 - \cos(\beta)\theta^2) \wedge \theta^2 = 0 \\ d\theta^2 &+ \sin(\beta)(w_1^2 + \cos(\beta)\theta^2) \wedge \theta^1 = 0 \\ d\theta^3 &= d\beta \wedge \theta^2 - \cos(\beta) w_2^1 \wedge \; w^1 + (1 + \sin^2(\beta))\theta^1 \wedge \theta^2 \end{aligned}$$

Therefore the connection form of S is given by

$$\theta_2^1 \;\; = \;\; \sin(\beta)(w_2^1 - \cos(\beta)\theta^2) \tag{5}$$

Differentiating e_3 at the basis (e_1, e_2), we have fundamental second forms coeficients

$$De_3 \;\; = \;\; \theta_3^1 e_1 + \theta_3^2 e_2$$

where

$$\begin{aligned} \theta_3^1 &= -\cos(\beta) w_2^1 - \sin^2(\beta)\theta^2 \\ \theta_3^2 &= d\beta + \theta^1 \end{aligned}$$

It follows from $d\theta^3 = 0$, that

$$w_2^1(e_2) \;\; = \;\; -\frac{\beta_1}{\cos\beta} - \frac{(1 + \sin^2\beta)}{\cos\beta} \tag{6}$$

where $d\beta(e_1) = \beta_1$.
The condition of minimality is equivalent to the following equation

$$\theta_1^3 \wedge \theta^2 - \theta_2^3 \wedge \theta^1 = 0$$

we have

$$w_2^1(e_1) \;\; = \tfrac{\beta_2}{\cos(\beta)} \tag{7}$$

where $d\beta(e_2) = \beta_2$.
It follows from (5), (6) and (7) that

$$\begin{aligned} \theta_2^1 &= \tan(\beta)(\beta_2\theta^1 - (\beta_1 + 2)\theta^2) \\ \theta_3^1 &= -\beta_2\theta^1 + (\beta_1 + 1)\theta^2 \\ \theta_3^2 &= (\beta_1 + 1)\theta^1 + \beta_2\theta^2 \end{aligned}$$

If J is the complex structure of S we have $Je_1 = e_2$ e $Je_2 = -e_1$.
Using J, the forms above simplify to:

$$\begin{aligned} \theta_2^1 &= \tan\beta(d\beta \circ J - 2\theta^2) \\ \theta_3^1 &= -d\beta \circ J + \theta^2 \\ \theta_3^2 &= d\beta + \theta^1 \end{aligned} \tag{8}$$

It follows from Gauss equation that

$$d\theta_1^2 = \theta^1 \wedge \theta^2 + \theta_1^3 \wedge \theta_2^3$$

We also have

$$d\theta_2^1 \;=\; (|\nabla\beta|^2 + 2\beta_1)\,(\theta^2 \wedge \theta^1) \tag{9}$$

and therefore

$$K \;=\; 1 - |\nabla\beta + e_1|^2$$

Differentiating θ_2^1, we have

$$\begin{aligned} d\theta_2^1 \;=\; & \sec^2(\beta)(|\nabla\beta|^2 + 2\beta_1)(\theta^2 \wedge \theta^1) \\ & +(\tan(\beta)\Delta(\beta) + 2\tan^2(\beta)(\beta_1 + 2))(\theta^2 \wedge \theta^1) \end{aligned} \tag{10}$$

Using (9) and (10), we obtain the following formula for the Laplacian of S

$$\Delta(\beta) \;=\; -\tan(\beta)((\beta_1 + 2)^2 + \beta_2^2) \tag{11}$$

Or

$$\Delta(\beta) \;=\; -\tan(\beta)|\nabla\beta + 2e_1|^2$$

Codazzi equations are

$$\begin{aligned} d\theta_1^3 + \theta_2^3 \wedge \theta_1^2 = 0 \\ d\theta_2^3 + \theta_1^3 \wedge \theta_2^1 = 0 \end{aligned}$$

The first equation gives (11) and the second equation is always verified.

4 Proof of the Theorem 1

Suppose that β is constant, it follows from (9) that $d\theta_1^2 = 0$ and, therefore,$K = 0$,ie., Gaussian curvature of S vanishes identically, hence $S \subset S^3$ is the Clifford torus, which prove the Theorem 1.

5 Proof of the Theorem 2

For $0 \leq \beta < \frac{\pi}{2}$, we have $\tan\beta \geq 0$, hence $\Delta(\beta) \leq 0$ and using that S is a compact surface, we conclude by Hopf's Lemma that β is constant, and therefore, $K = 0$ and S is the Clifford torus, which prove the Theorem 2.

6 Examples

6.1 Contact Angle of Clifford Torus in S^3

Consider the torus in S^3 defined by:

$$T^2 = \{(z_1, z_2) \in C^2 / z_1\bar{z_1} = \frac{1}{2}, z_2\bar{z}_2 = \frac{1}{2}\}$$

We consider the immersion:

$$f(u_1, u_2) = \frac{\sqrt{2}}{2}(e^{iu_1}, e^{iu_2})$$

$T(T^2)$ is generate by $\frac{\partial}{\partial u_1}$ and $\frac{\partial}{\partial u_2}$ it is means that:

$$a\frac{\partial}{\partial u_1} + b\frac{\partial}{\partial u_2} = \lambda z^{\perp}$$

using the condition above and the fact that $|\lambda| = 1$, we obtain:

$$\lambda \;=\; ie^{i(u_1+u_2)}$$

The unitary vector fields are:

$$\begin{cases} e_1 = ie^{i(u_1+u_2)}z^{\perp} \\ e_2 = iz \\ e_3 = e^{i\alpha}iz^{\perp} \end{cases}$$

The contact angle is the angle between e_2 and f_3 ,

$$\begin{aligned} cos(\beta) &= \langle e_2, f_3\rangle \\ &= 1 \end{aligned}$$

Therefore, the contact angle is:

$$\beta = 0$$

The fundamental second form at the basis (e_1, e_2) is:

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

6.2 Minimal surface in S^3 with non constant contact angle

Consider the surface described by:

$$\begin{cases} z_2 - \overline{z}_2 & = & 0 \\ (x_1)^2 + (y_1)^2 + (x_2)^2 + (y_2)^2 & = & 1 \end{cases}$$

We see that the unitary fields are:

$$\begin{cases} e_1 = \frac{1}{\sqrt{1-x_2^2}}(-x_1 x_2, -y_1 x_2, 1 - x_2^2, 0) \\ e_2 = \frac{1}{\sqrt{x_1^2+y_1^2}}(y_1, -x_1, 0, 0) \\ e_3 = (0, 0, 0, 1) \end{cases}$$

The contact angle is the angle between e_2 and f_3,

$$\begin{array}{rcl} cos(\beta) & = & \langle e_2, f_3 \rangle \\ & = & x_2 \end{array}$$

Therefore, the contact angle is:

$$\beta = arc\,cos\,(x_2)$$

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